THE LIMITS OF NESTED SUBCLASSES OF SEVERAL CLASSES OF INFINITELY DIVISIBLE DISTRIBUTIONS ARE IDENTICAL WITH THE CLOSURE OF THE CLASS OF STABLE DISTRIBUTIONS

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(*Running head* : Nested subclasses of infinitely divisible distributions)

Abstract. It is shown that the limits of the nested subclasses of five classes of infinitely divisible distributions on $\mathbb{R}^d$, which are the Jurek class, the Goldie–Steutel–Bondesson class, the class of selfdecomposable distributions, the Thorin class and the class of generalized type G distributions, are identical with the closure of the class of stable distributions. More general results are also given.

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1. Introduction

Subdivision of the class of infinitely divisible distributions on $\mathbb{R}^d$ has been an important subject since Urbanik's papers ([16], [17]). [7], [6], and [3] are some of many papers in this field. Among others, there are the Jurek class, the Goldie–Steutel–Bondesson class, the class of selfdecomposable distributions, the Thorin class, the class of type G distributions and their respective nested subclasses. Jurek ([6]) showed that two limits of the nested subclasses starting from the Jurek class and the class of selfdecomposable distributions are identical. It is also known (see [17] and

[10]) that the latter is the closure of the class of stable distributions, where the closure is taken under weak convergence and convolution.

In this paper, we treat five classes of infinitely divisible distributions on $\mathbb{R}^d$, all of which are characterized in terms of the radial components in the polar decomposition of the Lévy measures of infinitely divisible distributions, and the purpose of this paper is to show that the limits of the nested subclasses of these five classes are identical and equal to the closure of the class of stable distributions. In the course of the proof, we also give a more general theorem.

2. Preliminaries and the main result

Throughout the paper, $\mathcal{L}(X)$ denotes the law of an $\mathbb{R}^d$-valued random variable X and $\widehat{\mu}(z), z \in \mathbb{R}^d$, denotes the characteristic function of a probability distribution μ on $\mathbb{R}^d$. Also $I(\mathbb{R}^d)$ denotes the class of all infinitely divisible distributions on $\mathbb{R}^d$, $I_{\mathrm{sym}}(\mathbb{R}^d) = \{\mu \in I(\mathbb{R}^d) : \mu \text{ is symmetric on } \mathbb{R}^d\}$, $I_{\log}(\mathbb{R}^d) = \{\mu \in I(\mathbb{R}^d) : \int_{\mathbb{R}^d} \log^+ |x| \mu(dx) < \infty\}$ and $I_{\log^m}(\mathbb{R}^d) = \{\mu \in I(\mathbb{R}^d) : \int_{\mathbb{R}^d} (\log^+ |x|)^m \mu(dx) < \infty\}$. Further $S_\alpha(\mathbb{R}^d)$ denotes the class of α-stable distributions on $\mathbb{R}^d$ for $0 < \alpha \leqslant 2$ and $S(\mathbb{R}^d)$ denotes the class of stable distributions on $\mathbb{R}^d$. Let $C_\mu(z), z \in \mathbb{R}^d$, be the cumulant function of $\mu \in I(\mathbb{R}^d)$. That is, $C_\mu(z)$ is the unique continuous function with $C_\mu(0) = 0$ such that $\widehat{\mu}(z) = \exp(C_\mu(z)), z \in \mathbb{R}^d$.

We use the Lévy-Khintchine triplet (A, ν, γ) of $\mu \in I(\mathbb{R}^d)$ in the sense that

$$
\begin{aligned}
C_\mu(z) &= -2^{-1}\langle z, Az\rangle + \mathrm{i}\langle \gamma, z\rangle \\
&\qquad + \int_{\mathbb{R}^d} \left(e^{\mathrm{i}\langle z,x\rangle} - 1 - \mathrm{i}\langle z, x\rangle (1+|x|^2)^{-1}\right) \nu(dx),\ z \in \mathbb{R}^d,
\end{aligned}
\tag{2.1}
$$

where A is a symmetric nonnegative-definite $d \times d$ matrix, $\gamma \in \mathbb{R}^d$ and ν is a measure (called the Lévy measure) on $\mathbb{R}^d$ satisfying

$$\nu(\{0\}) = 0 \text{ and } \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(dx) < \infty.$$

The following is a basic result on the Lévy measure of $\mu \in I(\mathbb{R}^d)$.

Proposition 2.1. (Polar decomposition of Lévy measures.) ([9], [3]) *Let ν be the Lévy measure of the characteristic function of some $\mu \in I(\mathbb{R}^d)$ with $0 < \nu(\mathbb{R}^d) \leqslant \infty$. Then there exist a measure λ on $S = \{\xi \in \mathbb{R}^d : |\xi| = 1\}$ with $0 < \lambda(S) \leqslant \infty$ and a family $\{\nu_\xi : \xi \in S\}$ of measures on $(0, \infty)$ such that $\nu_\xi(B)$ is measurable in ξ for each*

$B \in \mathcal{B}((0,\infty))$, $0 < \nu_\xi((0,\infty)) \leqslant \infty$ *for each* $\xi \in S$, *and*

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)\nu_\xi(dr), \quad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}).$$

Here λ *and* $\{\nu_\xi\}$ *are uniquely determined by* ν *up to multiplication of measurable functions* $c(\xi)$ *and* $c(\xi)^{-1}$*, respectively, with* $0 < c(\xi) < \infty$*, and* ν_ξ *is called the radial component of* ν*. We call* $(\lambda(d\xi), \nu_\xi(dr))$ *a polar decomposition of the Lévy measure* $\nu \neq 0$*.*

Five classes in $I(\mathbb{R}^d)$ we are going to discuss in this paper are the following. As mentioned before, they are defined in terms of the radial component ν_ξ of Levy measures.

(1) Class $U(\mathbb{R}^d)$ (the Jurek class) :

$$\nu_\xi(dr) = \ell_\xi(r)dr,$$

where $\ell_\xi(r)$ is measurable in $\xi \in S$ and decreasing in $r \in (0,\infty)$. (Here and in what follows, we use the word "decreasing" in the non-strict sense.)

(2) Class $B(\mathbb{R}^d)$ (the Goldie–Steutel–Bondesson class) :

$$\nu_\xi(dr) = \ell_\xi(r)dr,$$

where $\ell_\xi(r)$ is measurable in $\xi \in S$ and completely monotone on $(0,\infty)$.

(3) Class $L(\mathbb{R}^d)$ (the class of selfdecomposable distributions) :

$$\nu_\xi(dr) = k_\xi(r)r^{-1}dr,$$

where $k_\xi(r)$ is measurable in $\xi \in S$ and decreasing on $(0,\infty)$.

(4) Class $T(\mathbb{R}^d)$ (the Thorin class) :

$$\nu_\xi(dr) = k_\xi(r)r^{-1}dr,$$

where $k_\xi(r)$ is measurable in $\xi \in S$ and completely monotone on $(0,\infty)$.

(5) Class $G(\mathbb{R}^d)$ (the class of generalized type G distributions) :

$$\nu_\xi(dr) = g_\xi(r^2)dr,$$

where $g_\xi(x)$ is measurable in $\xi \in S$ and completely monotone on $(0,\infty)$. (If $\mu \in G(\mathbb{R}^d)$ is symmetric, it is of type G distribution.)

From the definitions, it is trivial that

$$B(\mathbb{R}^d) \cup L(\mathbb{R}^d) \cup G(\mathbb{R}^d) \subset U(\mathbb{R}^d) \quad \text{and} \quad T(\mathbb{R}^d) \subset L(\mathbb{R}^d).$$

Also the same argument as in [2] shows that

$$T(\mathbb{R}^d) \subset B(\mathbb{R}^d) \subset G(\mathbb{R}^d).$$

For that, first note that

(1) the product of two completely monotone functions on $(0, \infty)$ is also completely monotone on $(0, \infty)$,

(2) $f(x) = x^{-\alpha}, \alpha > 0$, is completely monotone on $(0, \infty)$,

and

(3) if ϕ is a completely monotone function on $(0, \infty)$ and ψ is a nonnegative differentiable function on $(0, \infty)$ whose derivative is completely monotone, then the composition $\phi(\psi)$ is completely monotone on $(0, \infty)$, (see Corollary 2 in p. 441 of [4]). It follows from (1) and (2) that $T(\mathbb{R}^d) \subset B(\mathbb{R}^d)$. If we put $g_\xi(x) = l_\xi(x^{1/2})$, then by (3), we have that $B(\mathbb{R}^d) \subset G(\mathbb{R}^d)$. These inclusions are all strict, as shown below. For example, if we take a decreasing but not completely monotone function l_ξ, then we can see that $B(\mathbb{R}^d) \subsetneqq U(\mathbb{R}^d)$. Also, if we take a completely monotone function g_ξ which cannot be expressed as $g_\xi(x) = l_\xi(x^{1/2})$ with some completely monotone function l_ξ, then we see that $B(\mathbb{R}^d) \subsetneqq G(\mathbb{R}^d)$. Similar arguments work for all other cases.

Note that there are no relationships of inclusion between the classes $B(\mathbb{R}^d)$ and $L(\mathbb{R}^d)$, and the classes $G(\mathbb{R}^d)$ and $L(\mathbb{R}^d)$. Actually it is easy to see that $B(\mathbb{R}^d) \setminus L(\mathbb{R}^d) \neq \emptyset, L(\mathbb{R}^d) \setminus B(\mathbb{R}^d) \neq \emptyset, G(\mathbb{R}^d) \setminus L(\mathbb{R}^d) \neq \emptyset$, and $L(\mathbb{R}^d) \setminus G(\mathbb{R}^d) \neq \emptyset$.

These five classes are also characterized by mappings from infinitely divisible distributions to infinitely divisible distributions defined by the distributions of stochastic integrals with respect to Lévy processes. In what follows, $\{X_s^{(\mu)}\}$ stands for a Lévy process on $\mathbb{R}^d$ with $\mathcal{L}(X_1^{(\mu)}) = \mu$.

Definition 2.2. (1) ($\mathcal{U}$-mapping) ([5]) For $\mu \in I(\mathbb{R}^d)$,

$$\mathcal{U}(\mu) = \mathcal{L}\left(\int_0^1 s dX_s^{(\mu)}\right).$$

(2) (Υ-mapping) ([3]) For $\mu \in I(\mathbb{R}^d)$,

$$\Upsilon(\mu) = \mathcal{L}\left(\int_0^1 \log(s^{-1}) dX_s^{(\mu)}\right).$$

(3) (Φ-mapping) For $\mu \in I_{\log}(\mathbb{R}^d)$,

$$\Phi(\mu) = \mathcal{L}\left(\int_0^\infty e^{-s} dX_s^{(\mu)}\right).$$

(4) (Ψ-mapping) ([3]) Let $e(t) = \int_t^\infty e^{-u}u^{-1}du$, $t > 0$, and denote its inverse function by $e^*(s)$. For $\mu \in I_{\log}(\mathbb{R}^d)$,

$$\Psi(\mu) = \mathcal{L}\left(\int_0^\infty e^*(s)dX_s^{(\mu)}\right).$$

(5) ($\mathcal{G}$-mapping) Let $h(t) = \int_t^\infty e^{-u^2}du$, $t > 0$, and denote its inverse function by $h^*(s)$. For $\mu \in I(\mathbb{R}^d)$,

$$\mathcal{G}(\mu) = \mathcal{L}\left(\int_0^{\sqrt{\pi}/2} h^*(s)dX_s^{(\mu)}\right).$$

Remark 2.3. Letting $\mathfrak{D}$ denote the domain, we have $\mathfrak{D}(\mathcal{U}) = \mathfrak{D}(\Upsilon) = \mathfrak{D}(\mathcal{G}) = I(\mathbb{R}^d)$ and $\mathfrak{D}(\Phi) = \mathfrak{D}(\Psi) = I_{\log}(\mathbb{R}^d)$. Recall that $\mathfrak{D}(\Phi)$ and $\mathfrak{D}(\Psi)$ are defined as the class of $\mu \in I(\mathbb{R}^d)$ such that $\int_0^t e^{-s}dX_s^{(\mu)}$ and $\int_0^t e^*(s)dX_s^{(\mu)}$, respectively, are convergent in probability as $t \to \infty$. For two mappings Φ_1 and Φ_2, the composition $\Phi_2\Phi_1$ are defined as $\mathfrak{D}(\Phi_2\Phi_1) = \{\mu\colon \mu \in \mathfrak{D}(\Phi_1) \text{ and } \Phi_1(\mu) \in \mathfrak{D}(\Phi_2)\}$ and $(\Phi_2\Phi_1)(\mu) = \Phi_2(\Phi_1(\mu))$ for $\mu \in \mathfrak{D}(\Phi_2\Phi_1)$. It is known that $\Psi = \Upsilon\Phi = \Phi\Upsilon$ ([3]), where the equality of the domains is also implied.

The following are characterizations of the classes in the previous section in terms of the mappings above, or equivalently, in terms of stochastic integrals with respect to Lévy processes.

Theorem 2.4. (1) $U(\mathbb{R}^d) = \mathcal{U}(I(\mathbb{R}^d))$. ([5])
(2) $B(\mathbb{R}^d) = \Upsilon(I(\mathbb{R}^d))$. ([3])
(3) $L(\mathbb{R}^d) = \Phi(I_{\log}(\mathbb{R}^d))$. ([18] and others.)
(4) $T(\mathbb{R}^d) = \Psi(I_{\log}(\mathbb{R}^d))$. ([3])
(5) $G(\mathbb{R}^d) = \mathcal{G}(I(\mathbb{R}^d))$.

Remark 2.5. In [1], the equality (5) is proved within $I_{\mathrm{sym}}(\mathbb{R}^d)$. However, the same proof works for proving (5).

We now define the nested subclasses of the five classes above by iterating the respective mappings.

Let $U_0(\mathbb{R}^d) = U(\mathbb{R}^d)$, $B_0(\mathbb{R}^d) = B(\mathbb{R}^d)$, $L_0(\mathbb{R}^d) = L(\mathbb{R}^d)$, $T_0(\mathbb{R}^d) = T(\mathbb{R}^d)$ and $G_0(\mathbb{R}^d) = G(\mathbb{R}^d)$. In the following, the m-th power of a mapping denotes m times composition of the mapping, with the domain being the class of all μ for which the m-th power is definable.

Definition 2.6. For $m = 0, 1, 2, ...,$ let

(1) $U_m(\mathbb{R}^d) = \mathcal{U}^{m+1}(I(\mathbb{R}^d))$,
(2) $B_m(\mathbb{R}^d) = \Upsilon^{m+1}(I(\mathbb{R}^d))$,
(3) $L_m(\mathbb{R}^d) = \Phi^{m+1}(I_{\log^{m+1}}(\mathbb{R}^d))$,
(4) $T_m(\mathbb{R}^d) = \Psi^{m+1}(I_{\log^{m+1}}(\mathbb{R}^d))$,
and
(5) $G_m(\mathbb{R}^d) = \mathcal{G}^{m+1}(I(\mathbb{R}^d))$,
and let $U_\infty(\mathbb{R}^d) = \bigcap_{m=0}^\infty U_m(\mathbb{R}^d)$, $B_\infty(\mathbb{R}^d) = \bigcap_{m=0}^\infty B_m(\mathbb{R}^d)$, $L_\infty(\mathbb{R}^d) = \bigcap_{m=0}^\infty L_m(\mathbb{R}^d)$, $T_\infty(\mathbb{R}^d) = \bigcap_{m=0}^\infty T_m(\mathbb{R}^d)$, and $G_\infty(\mathbb{R}^d) = \bigcap_{m=0}^\infty G_m(\mathbb{R}^d)$.

Remark 2.7. (1) The original definition of $L_m(\mathbb{R}^d)$ in [17] and [10] is different from ours, but it is known that they are the same. See [15] or Lemma 4.1 of [3].
(2) We have $\mathfrak{D}(\Phi^{m+1}) = \mathfrak{D}(\Psi^{m+1}) = I_{\log^{m+1}}(\mathbb{R}^d)$. This will be shown in the proof of Lemma 3.8.
(3) $T_m(\mathbb{R}^d)$ here is different from that in [3], where $T_m(\mathbb{R}^d)$ denotes $\Upsilon(L_m(\mathbb{R}^d))$.
(4) $G_m(\mathbb{R}^d)$ here is different from that in [1], where everything is discussed within $I_{\mathrm{sym}}(\mathbb{R}^d)$.

As mentioned in the Introduction, the following are known.

Proposition 2.8. ([6]) $U_\infty(\mathbb{R}^d) = L_\infty(\mathbb{R}^d)$.

Remark 2.9. This is Corollary 7 of [6]. However, as Jurek's proof is not easy for us to follow, we will give in the last section of this paper an alternative proof for that $U_\infty(\mathbb{R}^d) \subset L_\infty(\mathbb{R}^d)$ which directly uses our polar decomposition of Lévy measures and shows that a representation of the Lévy measure of μ in $U_\infty(\mathbb{R}^d)$ is exactly the same as that of μ in $L_\infty(\mathbb{R}^d)$ shown in [10]. It is noted that our proof also depends on Jurek's basic idea.

Proposition 2.10. ([17] and [10]) $L_\infty(\mathbb{R}^d) = \overline{S(\mathbb{R}^d)}$, *where the closure is taken under weak convergence and convolution.*

Our main result in this paper is the following.

Theorem 2.11.

$$U_\infty(\mathbb{R}^d) = B_\infty(\mathbb{R}^d) = L_\infty(\mathbb{R}^d) = T_\infty(\mathbb{R}^d) = G_\infty(\mathbb{R}^d) = \overline{S(\mathbb{R}^d)}.$$

Except $T_\infty(\mathbb{R}^d)$, we will prove this theorem from a more general result (Theorem 3.4), where a sufficient condition for the limit of the nested subclasses of a class to be equal to $U_\infty(\mathbb{R}^d)$ is given. For $Y_\infty(\mathbb{R}^d)$, we will show that $T_\infty(\mathbb{R}^d) = L_\infty(\mathbb{R}^d)$.

3. Proof of the main theorem (Theorem 2.11)

In order to prove our main theorem (Theorem 2.11), we need several preparations.

Definition 3.1. A class M of distributions on $\mathbb{R}^d$ is said to be *completely closed in the strong sense* (c.c.s.s.), if $M \subset I(\mathbb{R}^d)$ and if the following are satisfied.
(1) It is closed under convolution.
(2) It is closed under weak convergence.
(3) If X is an $\mathbb{R}^d$-valued random variable with $\mathcal{L}(X) \in M$, then $\mathcal{L}(cX + b) \in M$ for any $c > 0$ and $b \in \mathbb{R}^d$.
(4) $\mu \in M$ implies $\mu^{s*} \in M$ for any $s > 0$, where μ^{s*} is the distribution with the characteristic function $(\widehat{\mu}(z))^s$.

Proposition 3.2. *Fix $0 < a < \infty$. Suppose that f is square integrable on $(0, a)$ and $\int_0^a f(s)ds \neq 0$. Define a mapping Φ_f by*

$$\Phi_f(\mu) = \mathcal{L}\left(\int_0^a f(s)dX_s^{(\mu)}\right).$$

Then the following are true.
(1) $\mathfrak{D}(\Phi_f) = I(\mathbb{R}^d)$.
(2) *For all $\mu \in I(\mathbb{R}^d)$, $\int_0^a |C_\mu(f(s)z)|ds < \infty$ and $C_{\Phi_f(\mu)}(z) = \int_0^a C_\mu(f(s)z)ds$.*
(3) *If M is c.c.s.s., then $\Phi_f(M) \subset M$.*
(4) *If M is c.c.s.s., then $\Phi_f(M)$ is also c.c.s.s.*

Proof. Define $\widetilde{f}(s)$ as $\widetilde{f}(s) = f(s)$ for $s \in (0, a)$ and $\widetilde{f}(s) = 0$ for $s \in [0, \infty) \setminus (0, a)$. Since $\widetilde{f}(s)$ is locally square integrable on $[0, \infty)$, $\int_B \widetilde{f}(s)dX_s^{(\mu)}$ is definable for all $\mu \in I(\mathbb{R}^d)$ and all bounded Borel sets B in $[0, \infty)$ by Proposition 3.4 of [13]. Then $\Phi_f(\mu)$ is the law of $\int_{(0,a)} \widetilde{f}(s)dX_s^{(\mu)}$. Hence (1) is true. (2) is a consequence of Proposition 2.17 of [13].

Proof of (3). Suppose that M is c.c.s.s. and $\mu \in M$. We recall the definition of $\int_B \widetilde{f}(s)dX_s^{(\mu)}$ in Sato [12] or [13]. A function $g(s)$ is called a simple function if $g(s) = \sum_{j=1}^n b_j 1_{B_j}(s)$ for some n, where $B_1, \ldots, B_n$ are disjoint Borel sets in $[0, \infty)$ and $b_1, \ldots, b_n \in \mathbb{R}$. For such a simple function we define $\int_B g(s)dX_s^{(\mu)} = \sum_{j=1}^n b_j X^{(\mu)}(B \cap B_j)$ for any bounded Borel set B in $[0, \infty)$, using the $\mathbb{R}^d$-valued independently scattered random measure $X^{(\mu)}$ induced by the process $X_s^{(\mu)}$. In our case the law of $Y = \int_B g(s)dX_s^{(\mu)}$ belongs to M, since

$$C_{\mathcal{L}(Y)}(z) = \sum_{j=1}^n \int_{B\cap B_j} C_\mu(b_j z)ds = \sum_{j=1}^n C_\mu(b_j z)\,\mathrm{Leb}(B \cap B_j),$$

where Leb denotes Lebesgue measure. Definability of $\int_B \widetilde{f}(s)dX_s^{(\mu)}$ mentioned in the proof of (1) means that there are simple functions $g_k(s)$, $k = 1, 2, \ldots$, such that $g_k(s) \to \widetilde{f}(s)$ a.e. as $k \to \infty$ and that, for all bounded Borel sets B, $\int_B g_k(s)dX_s^{(\mu)}$ converges in probability to $\int_B \widetilde{f}(s)dX_s^{(\mu)}$ as $k \to \infty$. Since M is closed under weak convergence, it follows that $\Phi_f(\mu) \in M$.

Proof of (4). Suppose that M is c.c.s.s. If μ_1 and μ_2 are in M, then $\Phi_f(\mu_1) * \Phi_f(\mu_2) = \Phi_f(\mu_1 * \mu_2) \in \Phi_f(M)$. Hence $\Phi_f(M)$ is closed under convolution. For $c > 0$ and $b \in \mathbb{R}^d$, we have

$$\int_0^a f(s)d(cX_s^{(\mu)} + bs) = c\int_0^a f(s)dX_s^{(\mu)} + b\int_0^a f(s)ds.$$

Since $\int_0^a f(s)ds \neq 0$, it follows from this that $\Phi_f(M)$ has property (3) of Definition 3.1. We have, for $t > 0$,

$$tC_{\Phi_f(\mu)}(z) = t\int_0^a C_\mu(f(s)z)ds = \int_0^a C_{\mu^{t*}}(f(s)z)ds.$$

Hence $\Phi_f(M)$ has property (4) of Definition 3.1. It remains to prove that $\Phi_f(M)$ is closed under weak convergence. We make use of the following fact for μ_n, $n = 1, 2\ldots$, and μ in $I(\mathbb{R}^d)$:

$$\text{if } \mu_n \to \mu, \text{ then } \Phi_f(\mu_n) \to \Phi_f(\mu). \tag{3.1}$$

To show this, let $\mu_n \to \mu$ and recall that

$$C_{\Phi_f(\mu_n)}(z) = \int_0^a C_{\mu_n}(f(s)z)ds,$$
$$C_{\Phi_f(\mu)}(z) = \int_0^a C_\mu(f(s)z)ds.$$

and that

$$C_{\mu_n}(f(s)z) \to C_\mu(f(s)z).$$

Hence it is enough to show the existence of an integrable function $h(s)$ on $(0, a)$ such that $\sup_n |C_{\mu_n}(f(s)z)| \leqslant c_z h(s)$ with constant c_z depending only on z and to use the dominated convergence theorem. Let (A_n, ν_n, γ_n) be the triplet of μ_n. Since μ_n is convergent, we have

$$\sup_n \operatorname{tr} A_n < \infty, \tag{3.2}$$

$$\sup_n \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu_n(dx) < \infty, \tag{3.3}$$

$$\sup_n |\gamma_n| < \infty. \tag{3.4}$$

We have

$$|C_\mu(z)| \leqslant \frac{1}{2}(\mathrm{tr}\ A)|z|^2 + |\gamma||z| + \int |g(z,x)|\nu(dx)$$

with $g(z,x) = e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle/(1+|x|^2)$. Hence

$$\begin{aligned}
|C_{\mu_n}(f(s)z)| &\leqslant \frac{1}{2}(\mathrm{tr}\ A_n)|f(s)z|^2 + |\gamma_n||f(s)z| \\
&\quad + \int_{\mathbb{R}^d} |g(z,f(s)x)|\nu_n(dx) + \int_{\mathbb{R}^d} |g(f(s)z,x) - g(z,f(s)x)|\nu_n(dx) \\
&= I_1 + I_2 + I_3 + I_4 \qquad \text{(say)}.
\end{aligned}$$

Let $c'_z, c''_z, \ldots$ denote constants depending on z. It follows from (3.2) and (3.4) that $I_1 + I_2 \leqslant c'_z(f(s)^2 + |f(s)|)$. Since $|g(z,x)| \leqslant c''_z|x|^2/(1+|x|^2)$, it follows from (3.3) that

$$\begin{aligned}
I_3 &\leqslant c''_z \int_{\mathbb{R}^d} \frac{|f(s)x|^2}{1+|f(s)x|^2}\nu_n(dx) \\
&\leqslant c''_z \left(f(s)^2 \int_{|x|\leqslant 1} |x|^2\nu_n(dx) + \int_{|x|>1} \nu_n(dx) \right) \\
&\leqslant c'''_z(f(s)^2+1).
\end{aligned}$$

Further, using

$$|g(uz,x) - g(z,ux)| \leqslant |z|\frac{|x|^3(|u|+|u|^3)}{(1+|x|^2)(1+|ux|^2)} \qquad \text{for } u \in \mathbb{R},$$

we obtain, with $u = f(s)$,

$$\begin{aligned}
I_4 &\leqslant |z| \int_{\mathbb{R}^d} \frac{|x|^3(|u|+|u|^3)}{(1+|x|^2)(1+|ux|^2)}\nu_n(dx) \\
&\leqslant |z| \int_{|x|\leqslant 1} \left(|x|^3|u| + \frac{|x|^2}{2}u^2 \right) \nu_n(dx) + |z| \int_{|x|>1} \left(\frac{|ux|}{1+|ux|^2} + \frac{|ux|}{1+|x|^2} \right) \nu_n(dx) \\
&\leqslant |z| \int_{|x|\leqslant 1} \left(|x|^3|u| + \frac{|x|^2}{2}u^2 \right) \nu_n(dx) + |z| \int_{|x|>1} \left(\frac{1}{2} + \frac{|u|}{2} \right) \nu_n(dx) \\
&\leqslant c''''_z(f(s)^2 + |f(s)| + 1)
\end{aligned}$$

from (3.3). Thus we get $h(s)$ as asserted. This proves (3.1). Now, let $\widetilde{\mu}_1, \widetilde{\mu}_2, \ldots$ be in $\Phi_f(M)$ and tend to $\widetilde{\mu}$. For each $\widetilde{\mu}_n$ we can find μ_n such that $\widetilde{\mu}_n = \Phi_f(\mu_n)$. Let $(\widetilde{A}_n, \widetilde{\nu}_n, \widetilde{\gamma}_n)$ and (A_n, ν_n, γ_n) be the triplets of $\widetilde{\mu}_n$ and μ_n, respectively. We claim that $\{\mu_n \colon n = 1, 2, \ldots\}$ is precompact, which is equivalent to (3.2), (3.3), (3.4), plus

$$\lim_{l\to\infty} \sup_n \int_{|x|>l} \nu_n(dx) = 0 \tag{3.5}$$

(see p. 13 of [3]). Since $\{\widetilde{\mu}_n\}$ is precompact, (3.2)–(3.5) hold for $(\widetilde{A}_n, \widetilde{\nu}_n, \widetilde{\gamma}_n)$ in place of (A_n, ν_n, γ_n). Recall that

$$\widetilde{A}_n = \left(\int_0^a f(s)^2 ds \right) A_n, \tag{3.6}$$

$$\widetilde{\nu}_n(B) = \int_0^a ds \int_{\mathbb{R}^d} 1_B(f(s)x)\nu_n(dx), \qquad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}), \tag{3.7}$$

$$\widetilde{\gamma}_n = \int_0^a f(s)ds \left(\gamma_n + \int_{\mathbb{R}^d} x \left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2} \right) \nu_n(dx) \right), \tag{3.8}$$

(see Proposition 2.6 of [14]). Hence we obtain (3.2). To see (3.3), note that

$$\int_{\mathbb{R}^d} (|x|^2 \wedge 1)\widetilde{\nu}_n(dx) = \int_0^a ds \int_{\mathbb{R}^d} (|f(s)x|^2 \wedge 1)\nu_n(dx)$$

and consider two cases separately: (1) there is $c > 0$ such that $|f(s)| \in \{0, c\}$ for a.e. $s \in (0, a)$; (2) otherwise. In case (1) we have

$$\begin{aligned} \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\widetilde{\nu}_n(dx) &= \int_{|f(s)|=c} ds \int_{\mathbb{R}^d} (|cx|^2 \wedge 1)\nu_n(dx) \\ &\geqslant (c^2 \wedge 1) \int_{|f(s)|=c} ds \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu_n(dx). \end{aligned}$$

In case (2), choosing $c > 0$ such that $\int_{|f(s)|\leqslant c} ds > 0$ and $\int_{|f(s)|>c} ds > 0$, we have

$$\begin{aligned} \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\widetilde{\nu}_n(dx) &= \int_0^a ds \int_{|f(s)x|\leqslant 1} |f(s)x|^2 \nu_n(dx) + \int_0^a ds \int_{|f(s)x|>1} \nu_n(dx) \\ &\geqslant \int_{|f(s)|\leqslant c} f(s)^2 ds \int_{|x|\leqslant 1/c} |x|^2 \nu_n(dx) + \int_{|f(s)|>c} ds \int_{|x|>1/c} \nu_n(dx). \end{aligned}$$

Hence we obtain (3.3) in any case. To prove (3.5), choose $c > 0$ with $\int_{|f(s)|>c} ds > 0$ and note that

$$\int_{|x|>l} \widetilde{\nu}_n(dx) = \int_0^a ds \int_{|f(s)x|>l} \nu_n(dx) \geqslant \int_{|f(s)|>c} ds \int_{|x|>l/c} \nu_n(dx).$$

In order to obtain (3.4), it suffices to show the boundedness of

$$\int_0^a f(s)ds \int_{\mathbb{R}^d} x \left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2} \right) \nu_n(dx)$$

since we have (3.8) and $\int_0^a f(s)ds \neq 0$. This boundedness is true because

$$\begin{aligned} &\int_0^a |f(s)|ds \int_{\mathbb{R}^d} |x| \left| \frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2} \right| \nu_n(dx) \\ &\quad \leqslant \int_0^a |f(s)|ds \int_{\mathbb{R}^d} \frac{|x|(|f(s)x|^2 + |x|^2)}{(1+|f(s)x|^2)(1+|x|^2)} \nu_n(dx) \\ &\quad \leqslant \int_0^a ds \left(\frac{f(s)^2}{2} + |f(s)| \right) \int_{|x|\leqslant 1} |x|^2 \nu_n(dx) + \int_0^a ds \left(\frac{|f(s)|}{2} + \frac{1}{2} \right) \int_{|x|>1} \nu_n(dx) \end{aligned}$$

$$\leqslant \text{const} \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu_n(dx).$$

Thus we have proved that $\{\mu_n\}$ is precompact. Therefore there exists a subsequence $\{\mu_{n_k}\}$ convergent to some $\mu \in M$. It follows from (3.1) that $\Phi_f(\mu_{n_k}) \to \Phi_f(\mu)$. Hence $\widetilde{\mu} = \Phi_f(\mu)$, concluding $\widetilde{\mu} \in \Phi_f(M)$. Therefore $\Phi_f(M)$ is closed under weak convergence, which completes the proof of Proposition 3.2. □

Remark 3.3. (1) Note that Proposition 3.2 can be applied to Υ- and $\mathcal{G}$-mappings, because in those mappings the upper limit of the stochastic integral is finite, f is square integrable, and $\int_0^a f(s)ds \neq 0$.

(2) Proposition 3.2 (4) is not necessarily true when $a = \infty$. Namely, there is a mapping Φ_f defined by $\Phi_f(\mu) = \mathcal{L}\left(\int_0^\infty f(s)dX_s^{(\mu)}\right)$ such that $\Phi_f(M \cap \mathfrak{D}(\Phi_f))$ is not closed under weak convergence for some M which is c.c.s.s. Indeed, let $\Phi_f = \Psi_\alpha$ with $0 < \alpha < 1$, which is defined similarly to Example 3.5 (3). Looking at Theorem 4.2 of [14], let μ_n be such that

$$C_{\mu_n}(z) = \int_S \lambda(d\xi) \int_0^\infty (e^{i\langle z, r\xi\rangle} - 1) r^{-\alpha-1} e^{-r/n} dr,$$

where λ is a finite nonzero measure on S. Then $\mu_n \in \Phi_f(\mathfrak{D}(\Phi_f))$ and μ_n tends to an α-stable distribution μ as $n \to \infty$, but $\mu \notin \Phi_f(\mathfrak{D}(\Phi_f))$ again by Theorem 4.2 of [14]. Thus $\Phi_f(\mathfrak{D}(\Phi_f))$ is not closed under weak convergence.

(3) However, it is known that when $\Phi_f = \Phi$, Proposition 3.2 (3) and (4) are true with $\Phi_f(M)$ replaced by $\Phi(M \cap \mathfrak{D}(\Phi))$, even if $a = \infty$. See Lemma 4.1 of [3]. In particular, $L_m(\mathbb{R}^d)$ is c.c.s.s. for $m = 0, 1, \ldots$.

We are now going to prove the following.

Theorem 3.4. *Let $0 < t_0 \leqslant \infty$. Let $p(u)$ be a positive decreasing function on $(0, t_0)$ such that $\int_0^{t_0} (1 + u^2) p(u) du < \infty$. Define $g(t) = \int_t^{t_0} p(u)du$ for $0 < t < t_0$ and $s_0 = g(0+) < \infty$. Let $t = f(s)$, $0 < s < s_0$, be the inverse function of $s = g(t)$, $0 < t < t_0$. Define*

$$\Phi_f(\mu) = \mathcal{L}\left(\int_0^{s_0} f(s)dX_s^{(\mu)}\right) \qquad \text{for } \mu \in \mathfrak{D}(\Phi_f).$$

Then,

(1) $\quad \mathfrak{D}(\Phi_f) = I(\mathbb{R}^d)$.

(2) $\quad I(\mathbb{R}^d) \supset \Phi_f(I(\mathbb{R}^d)) \supset \Phi_f^2(I(\mathbb{R}^d)) \supset \cdots$.

(3) $\quad \Phi_f^m(I(\mathbb{R}^d)) \subset U_{m-1}(\mathbb{R}^d)$ *for* $m = 1, 2, \ldots$.

(4) $\quad \Phi_f(S_\alpha(\mathbb{R}^d)) = S_\alpha(\mathbb{R}^d)$ *for* $0 < \alpha \leqslant 2$.

(5) $\quad \bigcap_{m=1}^{\infty} \Phi_f^m(I(\mathbb{R}^d)) = U_\infty(\mathbb{R}^d)$.

Example 3.5. The following are examples of Φ_f in Theorem 3.4.
(1) $\Phi_f = \Upsilon$ if $p(u) = e^{-u}$, $g(t) = e^{-t}$ with $t_0 = \infty$, and $f(s) = \log(s^{-1})$ with $s_0 = 1$.
(2) $\Phi_f = \mathcal{G}$ if $p(u) = e^{-u^2}$ with $t_0 = \infty$ and $s_0 = \sqrt{\pi}/2$.
(3) $\Phi_f = \Psi_\alpha$ with $-1 \leqslant \alpha < 0$ in [14] if $p(u) = u^{-\alpha-1}e^{-u}$ with $t_0 = \infty$ and $s_0 = \Gamma(-\alpha)$. Note that $\Psi_{-1} = \Upsilon$.
(4) $\Phi_f = \Phi_{\beta,\alpha}$ with $-1 \leqslant \alpha < 0$ and $\beta \leqslant \alpha - 1$ in [14] if

$$p(u) = (\Gamma(\alpha - \beta))^{-1}(1-u)^{\alpha-\beta-1}u^{-\alpha-1}$$

with $t_0 = 1$ and $s_0 = \Gamma(-\alpha)/\Gamma(-\beta)$.
(5) Φ_f with $f(s) = (1 + \alpha s)^{1/(-\alpha)}$, $-1 \leqslant \alpha < 0$, and $s_0 = 1/(-\alpha)$. This is a special case of (4) with $\beta = \alpha - 1$ and $g(t) = (1 - t^{-\alpha})/(-\alpha)$.
(6) Φ_f with $f(s) = 1 - (\Gamma(-\beta)s)^{1/(-\beta-1)}$, $\beta \leqslant -2$, and $s_0 = 1/\Gamma(-\beta)$. This is another special case of (4) with $\alpha = -1$ and $g(t) = (1-t)^{-\beta-1}/\Gamma(-\beta)$.
See p. 49 of [14] for (5) and (6). In particular, $\Phi_{-2,-1} = \mathcal{U}$, because in this case $p(u) = 1$, $g(t) = 1 - t$, $f(s) = 1 - s$, and $\int_0^1 C_\mu((1-s)z)ds = \int_0^1 C_\mu(sz)ds$.

In order to prove Theorem 3.4, we need two lemmas.

Lemma 3.6. *For* $j = 0, 1$ *let* $0 < s_j < \infty$ *and* $f_j(s)$ *be a square integrable function on* $(0, s_j)$. *Let*

$$\Phi_{f_j}(\mu) = \mathcal{L}\left(\int_0^{s_j} f_j(s) dX_s^{(\mu)}\right) \qquad \text{for } \mu \in \mathfrak{D}(\Phi_{f_j}) = I(\mathbb{R}^d).$$

Then

$$\Phi_{f_1}(\Phi_{f_0}(\mu)) = \Phi_{f_0}(\Phi_{f_1}(\mu)) \qquad \text{for } \mu \in I(\mathbb{R}^d). \tag{3.9}$$

Proof. We can check that

$$\int_0^{s_1} du \int_0^{s_0} |C_\mu(f_0(s)f_1(u)z)| ds < \infty \qquad \text{for } z \in \mathbb{R}^d, \tag{3.10}$$

because, as in the proof of Proposition 3.2 (4),

$$|C_\mu(f_0(s)f_1(u)z)| \leqslant c_z((f_0(s)f_1(u))^2 + |f_0(s)f_1(u)| + 1),$$

where c_z is a constant depending only on z. By virtue of (3.10), we can apply Fubini's theorem and

$$C_{\Phi_{f_1}(\Phi_{f_0}(\mu))}(z) = \int_0^{s_1} C_{\Phi_{f_0}(\mu)}(f_1(u)z)du = \int_0^{s_1} du \int_0^{s_0} C_\mu(f_0(s)f_1(u)z)ds$$

$$= \int_0^{s_0} ds \int_0^{s_1} C_\mu(f_1(u)f_0(s)z)du = \int_0^{s_0} C_{\Phi_{f_1}(\mu)}(f_0(s)z)ds = C_{\Phi_{f_0}(\Phi_{f_1}(\mu))}(z),$$

that is, (3.9) holds. □

Lemma 3.7. *Let $0 < s_0 < \infty$. Let $f(s)$ be a nonnegative, square integrable function on $(0, s_0)$ such that $\int_0^{s_0} f(s)ds > 0$. Then*

$$\Phi_f(S_\alpha(\mathbb{R}^d)) = S_\alpha(\mathbb{R}^d) \qquad \text{for } 0 < \alpha \leqslant 2.$$

Proof. A distribution μ is in $S_\alpha(\mathbb{R}^d)$ if and only if $\mu \in I(\mathbb{R}^d)$ and for any $c > 0$ there is $\gamma_c \in \mathbb{R}^d$ such that

$$\widehat{\mu}(cz) = \widehat{\mu}(z)^{c^\alpha} e^{i\langle \gamma_c, z\rangle}, \qquad z \in \mathbb{R}^d,$$

that is,

$$C_\mu(cz) = c^\alpha C_\mu(z) + i\langle \gamma_c, z\rangle.$$

For $c = 0$ this is trivially true with $\gamma_0 = 0$. If $\mu \in S_\alpha(\mathbb{R}^d)$, then

$$\begin{aligned} C_{\Phi_f(\mu)}(cz) &= \int_0^{s_0} C_\mu(f(s)cz)ds = \int_0^{s_0} c^\alpha C_\mu(f(s)z)ds + \int_0^{s_0} i\langle \gamma_c, f(s)z\rangle ds \\ &= c^\alpha C_{\Phi_f(\mu)}(z) + i \int_0^{s_0} f(s)ds \langle \gamma_c, z\rangle, \end{aligned}$$

which shows that $\Phi_f(\mu) \in S_\alpha(\mathbb{R}^d)$. Further, if $\mu \in S_\alpha(\mathbb{R}^d)$, then

$$C_{\Phi_f(\mu)}(z) = \int_0^{s_0} C_\mu(f(s)z)ds = \left(\int_0^{s_0} f(s)^\alpha ds\right) C_\mu(z) + i \left\langle \int_0^{s_0} \gamma_{f(s)} ds, z \right\rangle.$$

Recall (E 18.6 of [11]) that, if $\mu \in S_\alpha(\mathbb{R}^d)$, then there is $\tau \in \mathbb{R}^d$ such that

$$\gamma_c = \begin{cases} (c - c^\alpha)\tau & \text{for } \alpha \neq 1, \\ -c(\log c)\tau & \text{for } \alpha = 1, \end{cases}$$

which shows that $\int_0^{s_0} |\gamma_{f(s)}| ds < \infty$.

Conversely, suppose that $\widetilde{\mu} \in S_\alpha(\mathbb{R}^d)$ with $\widetilde{\gamma}_c$ in place of γ_c. Choose

$$\gamma = -\left(\int_0^{s_0} f(s)ds\right)^{-1} \left(\int_0^{s_0} f(s)^\alpha ds\right)^{-1} \int_0^{s_0} \widetilde{\gamma}_{f(s)} ds.$$

Let $\mu \in I(\mathbb{R}^d)$ be such that

$$C_\mu(z) = \left(\int_0^{s_0} f(s)^\alpha ds\right)^{-1} C_{\widetilde{\mu}}(z) + i\langle \gamma, z\rangle.$$

Then $\mu \in S_\alpha(\mathbb{R}^d)$ and

$$C_{\Phi_f(\mu)}(z) = \int_0^{s_0} C_\mu(f(s)z)ds$$

$$
\begin{aligned}
&= \left(\int_0^{s_0} f(s)^\alpha ds\right)^{-1} \int_0^{s_0} C_{\widetilde{\mu}}(f(s)z)ds + i\int_0^{s_0} \langle \gamma, f(s)z\rangle ds \\
&= C_{\widetilde{\mu}}(z) + i\left(\int_0^{s_0} f(s)^\alpha ds\right)^{-1} \int_0^{s_0} \langle \widetilde{\gamma}_{f(s)}, z\rangle ds + i\int_0^{s_0} f(s)ds\langle \gamma, z\rangle \\
&= C_{\widetilde{\mu}}(z),
\end{aligned}
$$

and hence $\Phi_f(\mu) = \widetilde{\mu}$. This completes the proof. □

Proof of Theorem 3.4. In the following, we write I for $I(\mathbb{R}^d)$ for simplicity.

(1) Since $s_0 < \infty$ and since

$$
\int_0^{s_0} f(s)^2 ds = \int_0^{t_0} t^2 p(t)dt < \infty,
$$

we have $\mathfrak{D}(\Phi_f) = I$. See Proposition 3.2 (1).

(2) It follows from $\Phi_f(I) \subset I$ that $\Phi_f^2(I) \subset \Phi_f(I)$. Then, $\Phi_f^3(I) \subset \Phi_f^2(I)$, and so on.

(3) Let $\widetilde{\mu} = \Phi_f(\mu)$. Let $\widetilde{\nu}$ and ν be the Lévy measures of $\widetilde{\mu}$ and μ, respectively. Let $(\lambda(d\xi), \nu_\xi(dr))$ be a polar decomposition of ν. We know

$$
\widetilde{\nu}(B) = \int_0^{s_0} ds \int_{\mathbb{R}^d} 1_B(f(s)x)\nu(dx) = \int_0^{t_0} p(t)dt \int_{\mathbb{R}^d} 1_B(tx)\nu(dx)
$$

for $B \in \mathcal{B}(\mathbb{R}^d)$. If $B = \{r\xi \colon \xi \in D,\ r \in (s,\infty)\}$ with $D \in \mathcal{B}(S)$ and $s > 0$, then

$$
\begin{aligned}
\widetilde{\nu}(B) &= \int_0^{t_0} p(t)dt \int_D \lambda(d\xi) \int_{s/t}^\infty \nu_\xi(dr) = \int_D \lambda(d\xi) \int_{s/t_0}^\infty \nu_\xi(dr) \int_{s/r}^{t_0} p(t)dt \\
&= \int_D \lambda(d\xi) \int_{s/t_0}^\infty r^{-1}\nu_\xi(dr) \int_s^{rt_0} p(u/r)du \\
&= \int_D \lambda(d\xi) \int_s^\infty du \int_{u/t_0}^\infty p(u/r)r^{-1}\nu_\xi(dr).
\end{aligned}
$$

Hence, letting $\widetilde{\lambda} = \lambda$ and

$$
\widetilde{l}_\xi(u) = \int_{u/t_0}^\infty p(u/r)r^{-1}\nu_\xi(dr),
$$

we obtain a polar decomposition $(\widetilde{\lambda}(d\xi), \widetilde{l}_\xi(u)du)$ of $\widetilde{\nu}$. Since p is decreasing, $\widetilde{l}_\xi(u)$ is decreasing in u. Therefore $\widetilde{\mu} \in U(\mathbb{R}^d) = U_0(\mathbb{R}^d)$. Hence $\Phi_f(I) \subset U_0(\mathbb{R}^d) = \mathcal{U}(I)$. This proves (3) for $m = 1$.

If $\Phi_f^m(I) \subset \mathcal{U}^m(I) = U_{m-1}(\mathbb{R}^d)$, then, using Lemma 3.6, we get

$$
\Phi_f^{m+1}(I) \subset \Phi_f(\mathcal{U}^m(I)) = \mathcal{U}^m(\Phi_f(I)) \subset \mathcal{U}^{m+1}(I) = U_m(\mathbb{R}^d).
$$

This completes the induction argument.

(4) Apply Lemma 3.7.

(5) It follows from (3) that

$$\bigcap_{m=1}^{\infty} \Phi_f^m(I) \subset \bigcap_{m=0}^{\infty} U_m(\mathbb{R}^d) = U_\infty(\mathbb{R}^d).$$

On the other hand, it follows from (4) that

$$S_\alpha(\mathbb{R}^d) = \Phi_f^m(S_\alpha(\mathbb{R}^d)) \subset \Phi_f^m(I).$$

Hence $S(\mathbb{R}^d) = \bigcup_{0<\alpha\leqslant 2} S_\alpha(\mathbb{R}^d) \subset \bigcap_{m=1}^{\infty} \Phi_f^m(I)$. Use Proposition 3.2 (4) to show that $\Phi_f^m(I)$ is c.c.s.s. Then we see that $\overline{S(\mathbb{R}^d)} \subset \bigcap_{m=1}^{\infty} \Phi_f^m(I)$. Since $\overline{S(\mathbb{R}^d)} = L_\infty(\mathbb{R}^d)$ (Proposition 2.10) and $L_\infty(\mathbb{R}^d) = U_\infty(\mathbb{R}^d)$ (Proposition 2.8), the proof of (5) is complete. □

We need one more lemma.

Lemma 3.8. *$T_m(\mathbb{R}^d)$ is c.c.s.s. for $m = 0, 1, \ldots$.*

Proof. We first show

(1) $\mu \in I_{\log^m}(\mathbb{R}^d)$ if and only if $\Upsilon(\mu) \in I_{\log^m}(\mathbb{R}^d)$

and

(2) $\mu \in I_{\log^{m+1}}(\mathbb{R}^d)$ if and only if $\mu \in I_{\log}(\mathbb{R}^d)$ and $\Phi(\mu) \in I_{\log^m}(\mathbb{R}^d)$.

Let us prove (1). If $\int_{|y|>1}(\log|y|)^m \nu_\mu(dy) < \infty$, then

$$\begin{aligned}
\int_{|x|>1} (\log|x|)^m \nu_{\Upsilon(\mu)}(dx) &= \int_{|x|>1} (\log|x|)^m \int_0^\infty \nu_\mu(s^{-1}dx)e^{-s}ds \\
&= \int_{\mathbb{R}^d} \nu_\mu(dy) \int_0^\infty (\log|sy|)^m e^{-s} 1_{\{|sy|>1\}} ds \\
&= \int_{|y|>0} \nu_\mu(dy) \int_{1/|y|}^\infty (\log|sy|)^m e^{-s} ds \\
&= \int_{|y|>0} \nu_\mu(dy) \int_{1/|y|}^\infty \sum_{n=0}^{m} \binom{m}{n} (\log s)^n (\log|y|)^{m-n} e^{-s} ds \\
&= \sum_{n=0}^{m} \binom{m}{n} \int_{|y|>1} (\log|y|)^{m-n} \nu_\mu(dy) \int_{1/|y|}^\infty (\log s)^n e^{-s} ds + \text{finite term},
\end{aligned}$$

which implies $\int_{|x|>1}(\log|x|)^m \nu_{\Upsilon(\mu)}(dx) < \infty$. Here we have used that

$$\int_{1/|y|}^\infty (\log s)^n e^{-s} ds \sim (\log(1/|y|))^n e^{-1/|y|}, \qquad |y| \to 0,$$

and that $\int_0^\infty (\log s)^n e^{-s} ds$ is finite. Conversely, if $m \geqslant 1$ and $\int_{|x|>1}(\log|x|)^m \nu_{\Upsilon(\mu)}(dx) < \infty$ and if (1) is true for $m-1$ in place of m, then $\int_{|y|>1}(\log|y|)^j \nu_\mu(dy) < \infty$ for

$j = 0, \dots, m-1$, and the equalities above show that $\int_{|y|>1} (\log |y|)^m \nu_\mu(dy) < \infty$. As (1) is trivially true for $m = 0$, we see that (1) is true for all m.

Assertion (2) follows from

$$\begin{aligned}\int_{|x|>1} (\log |x|)^m \nu_{\Phi(\mu)}(dx) &= \int_{|x|>1} (\log |x|)^m \int_0^\infty \nu_\mu(e^s dx) ds \\ &= \int_{\mathbb{R}^d} \nu_\mu(dy) \int_0^\infty (\log |e^{-s} y|)^m 1_{\{|e^{-s}y|>1\}} ds \\ &= \int_{|y|>1} \nu_\mu(dy) \int_0^{\log |y|} (\log |y| - s)^m ds \\ &= (m+1)^{-1} \int_{|y|>1} (\log |y|)^{m+1} \nu_\mu(dy).\end{aligned}$$

It follows from (2) that $\mathfrak{D}(\Phi^m) = I_{\log^m}(\mathbb{R}^d)$. Since $\Psi = \Phi\Upsilon = \Upsilon\Phi$, it follows from (1) and (2) that
(3) $\mu \in I_{\log^{m+1}}(\mathbb{R}^d)$ if and only if $\mu \in I_{\log}(\mathbb{R}^d)$ and $\Psi(\mu) \in I_{\log^m}(\mathbb{R}^d)$.
Hence $\mathfrak{D}(\Psi^m) = I_{\log^m}(\mathbb{R}^d)$. Thus, we have

$$T_m(\mathbb{R}^d) = \Psi^{m+1}(I_{\log^{m+1}}(\mathbb{R}^d)) = (\Upsilon^{m+1}\Phi^{m+1})(I_{\log^{m+1}}(\mathbb{R}^d)),$$

that is,

$$T_m(\mathbb{R}^d) = \Upsilon^{m+1}(L_m(\mathbb{R}^d)). \tag{3.11}$$

Since $L_m(\mathbb{R}^d)$ is c.c.s.s. by Remark 3.3 (3), it follows from Proposition 3.2 (4) that $T_m(\mathbb{R}^d)$ is c.c.s.s. □

We are now ready to prove Theorem 2.11.

Proof of Theorem 2.11. We have already seen that $U_\infty(\mathbb{R}^d) = L_\infty(\mathbb{R}^d) = \overline{S(\mathbb{R}^d)}$. Since Υ- and $\mathcal{G}$-mappings are examples of Φ_f in Theorem 3.4, it follows from Theorem 3.4 (5) that $B_\infty(\mathbb{R}^d) = G_\infty(\mathbb{R}^d) = U_\infty(\mathbb{R}^d) = \overline{S(\mathbb{R}^d)}$. It remains to show that $T_\infty(\mathbb{R}^d) = \overline{S(\mathbb{R}^d)}$. It follows from (3.11), Proposition 3.2 (3), (4), and Remark 3.3 (3) that $T_m(\mathbb{R}^d) = \Upsilon^{m+1}(L_m(\mathbb{R}^d)) \subset L_m(\mathbb{R}^d)$. Hence

$$T_\infty(\mathbb{R}^d) \subset L_\infty(\mathbb{R}^d). \tag{3.12}$$

It follows from (3.11) and Lemma 3.7 that $T_m(\mathbb{R}^d) = \Upsilon^{m+1}(L_m(\mathbb{R}^d)) \supset \Upsilon^{m+1}(S(\mathbb{R}^d)) = S(\mathbb{R}^d)$. Hence we get $T_m(\mathbb{R}^d) \supset \overline{S(\mathbb{R}^d)}$ from Lemma 3.8. Therefore

$$T_\infty(\mathbb{R}^d) \supset \overline{S(\mathbb{R}^d)} = L_\infty(\mathbb{R}^d). \tag{3.13}$$

Thus, (3.12) and (3.13) imply that $T_\infty(\mathbb{R}^d) = L_\infty(\mathbb{R}^d)$, which completes the proof of Theorem 2.11. □

4. Proof of Proposition 2.8

As we announced, we give here our proof of Proposition 2.8. We start with the following fact.

Proposition 4.1. *Let $\mu \in U_0(\mathbb{R}^d)$ with Lévy measure $\nu^\mu \neq 0$. Let $\mu = \mathcal{U}(\rho)$ and let $(\lambda^\rho(d\xi), \nu_\xi^\rho(dr))$ be a polar decomposition of the Lévy measure ν^ρ of ρ. Let $\lambda^\mu = \lambda^\rho$ and let*

$$l_\xi^\mu(r) = \int_r^\infty s^{-1}\nu_\xi^\rho(ds), \qquad r > 0, \tag{4.1}$$

$$\nu_\xi^\mu(dr) = l_\xi^\mu(r)dr, \qquad r > 0, \tag{4.2}$$

$$h_\xi^\rho(u) = e^{-u}\nu_\xi^\rho((e^u, \infty)), \qquad u \in \mathbb{R}, \tag{4.3}$$

$$h_\xi^\mu(u) = e^{-u}\nu_\xi^\mu((e^u, \infty)), \qquad u \in \mathbb{R}. \tag{4.4}$$

Then the following are true.
(1) $(\lambda^\mu(d\xi), \nu_\xi^\mu(dr))$ is a polar decomposition of ν^μ.
(2) $h_\xi^\mu(u)$ is absolutely continuous on $\mathbb{R}$ and

$$-\frac{d}{du}h_\xi^\mu(u) = h_\xi^\rho(u), \qquad \textit{for } \text{a.e. } u \in \mathbb{R},$$

where $\frac{d}{du}$ denotes Radon–Nikodým derivative.

Proof. It follows from Definition 2.2 (1) that

$$\nu^\mu(B) = \int_0^1 \nu^\rho(t^{-1}B)dt, \qquad B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}).$$

We have $\nu^\rho \neq 0$, since $\nu^\mu \neq 0$. Let $v > 0$ and $D \in \mathcal{B}(S)$. For $B = (v, \infty)D = \{x = r\xi \colon \xi \in D,\, r > v\}$, notice that

$$\begin{aligned}
\nu^\mu(B) &= \int_0^1 dt \int_D \lambda^\rho(d\xi) \int_{t^{-1}v}^\infty \nu_\xi^\rho(dr) = \int_D \lambda^\rho(d\xi) \int_v^\infty \nu_\xi^\rho(dr) \int_{v/r}^1 dt \\
&= \int_D \lambda^\rho(d\xi) \int_v^\infty \nu_\xi^\rho(dr) \int_v^r r^{-1}du = \int_D \lambda^\rho(d\xi) \int_v^\infty du \int_u^\infty r^{-1}\nu_\xi^\rho(dr) \\
&= \int_S \lambda^\mu(d\xi) \int_0^\infty 1_{(v,\infty)D}(r\xi) l_\xi^\mu(r)dr,
\end{aligned}$$

by (4.1). Thus for a general $B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$, we have

$$\nu^\mu(B) = \int_S \lambda_\xi^\mu(d\xi) \int_0^\infty 1_B(r\xi) l_\xi^\mu(r)dr.$$

Hence (1) is true. Since

$$h_\xi^\mu(u) = e^{-u}\int_{e^u}^\infty l_\xi^\mu(s)ds, \qquad u \in \mathbb{R},$$

absolute continuity of $h^\mu_\xi(u)$ is obvious. We have, for a.e. $u \in \mathbb{R}$,

$$\begin{aligned}
-\frac{d}{du} h^\mu_\xi(u) &= e^{-u} \int_{e^u}^\infty l^\mu_\xi(s) ds + l^\mu_\xi(e^u) \\
&= e^{-u} \int_{e^u}^\infty ds \int_s^\infty r^{-1} \nu^\rho_\xi(dr) + \int_{e^u}^\infty r^{-1} \nu^\rho_\xi(dr) \\
&= e^{-u} \int_{e^u}^\infty r^{-1} \nu^\rho_\xi(dr) \int_{e^u}^r ds + \int_{e^u}^\infty r^{-1} \nu^\rho_\xi(dr) \\
&= \int_{e^u}^\infty r^{-1} \nu^\rho_\xi(dr) \left(e^{-u}(r - e^u) + 1 \right) \\
&= e^{-u} \nu^\rho_\xi((e^u, \infty)) = h^\rho_\xi(u).
\end{aligned}$$

This completes the proof of (2). □

The next two propositions give us some properties of $\mu \in U_m(\mathbb{R}^d)$ for $m \in \{0, 1, 2, ..., \infty\}$.

Proposition 4.2. *Let $m \in \{0, 1, \ldots\}$. Suppose that $\mu \in U_m(\mathbb{R}^d)$ with Lévy measure $\nu \neq 0$. Let $(\lambda(d\xi), \nu_\xi(dr))$ be a polar decomposition of ν. Let*

$$h_\xi(u) = e^{-u} \nu_\xi((e^u, \infty)), \qquad u \in \mathbb{R}. \tag{4.5}$$

Then, for λ-a.e. ξ, $h_\xi(u)$ is m times differentiable on $\mathbb{R}$ and $(d/du)^m h_\xi(u)$ is absolutely continuous on $\mathbb{R}$. Moreover,

$$\left(-\frac{d}{du}\right)^j h_\xi(u) \geqslant 0 \qquad \textit{for all } u \in \mathbb{R} \quad \textit{for } j = 0, 1, \ldots, m \tag{4.6}$$

and

$$\left(-\frac{d}{du}\right)^{m+1} h_\xi(u) \geqslant 0 \qquad \textit{for } \text{ a.e. } u \in \mathbb{R}. \tag{4.7}$$

Proof. Step 1. The case $m = 0$. We have $\mu \in U_0(\mathbb{R}^d) = \mathcal{U}(I(\mathbb{R}^d))$. Hence there is ρ such that $\mu = \mathcal{U}(\rho)$. We have $\nu^\rho \neq 0$ for the Lévy measure ν^ρ of ρ. Let $(\lambda^\rho(d\xi), \nu^\rho_\xi(dr))$ be a polar decomposition of ν^ρ. Then Proposition 4.1 gives a polar decomposition $(\lambda^\mu(d\xi), \nu^\mu_\xi(dr))$ of the Lévy measure ν of μ. On the other hand, ν has a polar decomposition $(\lambda(d\xi), \nu_\xi(dr))$. Hence it follows from Proposition 2.1 that there is a measurable function $0 < c(\xi) < \infty$ such that $\lambda(d\xi) = c(\xi)\lambda^\mu(d\xi)$ and, for λ-a.e. ξ, $\nu_\xi(dr) = c(\xi)^{-1}\nu^\mu_\xi(dr)$. Hence $h_\xi(u) = c(\xi)^{-1} h^\mu_\xi(u)$ for λ-a.e. ξ. Thus, Proposition 4.1 shows that, for λ-a.e. ξ, $h_\xi(u)$ is absolutely continuous and $(-d/du)h_\xi(u) = c(\xi)^{-1} h^\rho_\xi(u) \geqslant 0$.

Step 2. Suppose that the statement is true for m. Suppose that $\mu \in U_{m+1}(\mathbb{R}^d) = \mathcal{U}^{m+2}(I(\mathbb{R}^d))$ with Lévy measure $\nu \neq 0$. Then there is $\rho \in \mathcal{U}^{m+1}(I(\mathbb{R}^d))$ such that

$\mu = \mathcal{U}(\rho)$. The same argument as in Step 1 shows that, for λ-a.e. ξ, $h_\xi(u)$ is absolutely continuous and $(-d/du)h_\xi(u) = c(\xi)^{-1}h_\xi^\rho(u)$ for a.e. $u > 0$. Moreover $h_\xi(u)$ is differentiable and this equality holds for all $u > 0$, because $h_\xi^\rho(u)$ is continuous since $\rho \in U_m(\mathbb{R}^d) \subset U_0(\mathbb{R}^d)$. Now, using the induction hypothesis, we see that $h_\xi^\rho(u)$ satisfies our assertion with m replaced by $m+1$. □

Proposition 4.3. *Suppose that $\mu \in U_\infty(\mathbb{R}^d)$ with Lévy measure $\nu \neq 0$. Let $(\lambda(d\xi), \nu_\xi(dr))$ be a polar decomposition of ν. Define $h_\xi(u)$ by (4.5). Then, for λ-a.e. ξ, $h_\xi(u)$ is completely monotone on $\mathbb{R}$.*

Proof. This is clear from Proposition 4.2. □

Now we use Bernstein's theorem and the representation theorem for $L_\infty(\mathbb{R}^d)$.

Proposition 4.4. $U_\infty(\mathbb{R}^d) \subset L_\infty(\mathbb{R}^d)$.

Proof. Let $\mu \in U_\infty(\mathbb{R}^d)$. If μ is Gaussian, then obviously $\mu \in L_\infty(\mathbb{R}^d)$. Suppose that μ has Lévy measure $\nu \neq 0$. Choose the polar decomposition $(\lambda(d\xi), \nu_\xi(dr))$ of ν such that

$$\int_0^\infty (r^2 \wedge 1)\nu_\xi(dr) = c = \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(dx), \qquad \xi \in S$$

and $\lambda(d\xi)$ is a probability measure. Let $h_\xi(u) = e^{-u}\nu_\xi((e^u, \infty))$. Then it follows from Proposition 4.3 that, for λ-a.e. ξ, $h_\xi(u)$ is completely monotone on $\mathbb{R}$. For $a \in \mathbb{R}$, $h_\xi(a+u)$ is a completely monotone function of $u > 0$. Hence by Bernstein's theorem there is a unique measure H_ξ^a on $[0, \infty)$ such that

$$h_\xi(a+u) = \int_{[0,\infty)} e^{-uv} H_\xi^a(dv), \qquad u > 0.$$

If $a_1 < a_2$, then

$$h_\xi(a_2+u) = h_\xi(a_1 + (a_2 - a_1) + u) = \int_{[0,\infty)} e^{-(a_2-a_1+u)v} H_\xi^{a_1}(dv)$$

for $u > 0$ and, by the uniqueness, we have

$$e^{-(a_2-a_1)v} H_\xi^{a_1}(dv) = H_\xi^{a_2}(dv).$$

Hence $e^{av}H_\xi^a(dv)$ is independent of a. Write this measure as $H_\xi(dv)$. Then

$$h_\xi(a+u) = \int_{[0,\infty)} e^{-uv} e^{-av} H_\xi(dv)$$

for all $a \in \mathbb{R}$ and $u > 0$. It follows that

$$h_\xi(u) = \int_{[0,\infty)} e^{-uv} H_\xi(dv), \qquad u \in \mathbb{R}. \tag{4.8}$$

As in p. 218 of [10] or p. 17 of [8], we can prove that, for any $B \in \mathcal{B}([0,\infty))$, $H_\xi(B)$ is measurable in ξ. The identity (4.8) can be written as

$$\nu_\xi((r,\infty)) = \int_{[0,\infty)} r^{1-v} H_\xi(dv), \qquad r > 0. \tag{4.9}$$

Since the left-hand side tends to 0 as $r \to \infty$, we obtain $H_\xi(\,[0,1]\,) = 0$. Now we get

$$-\frac{d}{dr}(\nu_\xi((r,\infty))) = \int_{(1,\infty)} (v-1) r^{-v} H_\xi(dv), \qquad \text{a.e. } r > 0.$$

Since

$$\infty > \int_0^1 r^2 \nu_\xi(dr) = \int_0^1 r^2 dr \int_{(1,\infty)} (v-1) r^{-v} H_\xi(dv) = \int_{(1,\infty)} (v-1) H_\xi(dv) \int_0^1 r^{2-v} dr,$$

we obtain $H_\xi(\,[3,\infty)\,) = 0$. Define

$$\Gamma_\xi(E) = \int_{(1,3)} 1_E(v-1)\,(v-1) H_\xi(dv), \qquad E \in \mathcal{B}([0,\infty)).$$

Then Γ_ξ is concentrated on $(0,2)$ and

$$\int_{(0,2)} f(\alpha)\Gamma_\xi(d\alpha) = \int_{(1,3)} f(v-1)\,(v-1) H_\xi(dv), \qquad \text{for all measurable } f \geqslant 0.$$

Now,

$$\frac{d}{dr}(\nu_\xi((r,\infty))) = \int_{(1,3)} (v-1) r^{-(v-1)-1} H_\xi(dv) = \int_{(0,2)} r^{-\alpha-1}\Gamma_\xi(d\alpha),$$

$$\int_{(0,1]} r^2 \nu_\xi(dr) = \int_{(0,2)} \frac{\Gamma_\xi(d\alpha)}{2-\alpha},$$

$$\int_{(1,\infty)} \nu_\xi(dr) = \int_{(0,2)} \frac{\Gamma_\xi(d\alpha)}{\alpha}.$$

Hence

$$\int_{(0,2)} \left(\frac{1}{\alpha} + \frac{1}{2-\alpha}\right) \Gamma_\xi(d\alpha) = c.$$

We can find a finite measure Γ on $(0,2)$ and probability measures λ_α on S such that $\lambda_\alpha(D)$ is measurable in α for any $D \in \mathcal{B}(S)$ and $\Gamma(d\alpha)\lambda_\alpha(d\xi) = \lambda(d\xi)\Gamma_\xi(d\alpha)$. Thus,

$$\int_{(0,2)} \left(\frac{1}{\alpha} + \frac{1}{2-\alpha}\right) \Gamma(d\alpha) = c$$

and, for any $B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$,

$$\begin{aligned}\nu(B) &= \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)\nu_\xi(dr) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) dr \int_{(0,2)} r^{-\alpha-1}\Gamma_\xi(d\alpha) \\ &= \int_{(0,2)} \Gamma(d\alpha) \int_S \lambda_\alpha(d\xi) \int_0^\infty 1_B(r\xi) r^{-\alpha-1} dr.\end{aligned}$$

This is exactly the form of the Lévy measure in Theorem 22 of [8] (originally Theorem 3.4 of [10]). This shows that $\mu \in L_\infty(\mathbb{R}^d)$. The proof is completed. $\square$

Finally we have the following.

Proposition 4.5. $U_\infty(\mathbb{R}^d) = L_\infty(\mathbb{R}^d)$.

Proof. It remains to prove that $U_\infty(\mathbb{R}^d) \supset L_\infty(\mathbb{R}^d)$, but this is concluded from that $U_m(\mathbb{R}^d) \supset L_m(\mathbb{R}^d)$ for each $m \geq 1$, which is shown in [6]. $\square$

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